Non-Archimedean second main theorems and applications

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Definition

Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a non-constant analytic curve, $\tilde{f} = (f_0, \ldots, f_N)$ be a reduced representative of f.
Let

$$||\tilde{f}(z)|| := \max\{|f_0(z)|, \ldots, |f_N(z)|\}.$$

The **Nevanlinna characteristic function** of f is

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log ||\tilde{f}(re^{i\theta})|| d\theta.$$

Definition

The **proximity function** of a homogeneous polynomial Q of degree d respect to the map f is defined by

$$m_f(r,Q) := \int_0^{2\pi} \log^+ \frac{||\tilde{f}(re^{i\theta})||^d}{|Q(\tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi}.$$

The **counting function** is

$$N_f(r,Q) = \int_0^r \frac{n_f(t,Q) - n_f(0,Q)}{t} dt + n_f(0,Q) \log r,$$

where

$$n_f(r,Q) := \#\{z \mid |z| < r, Q \circ f(z) = 0\}, \text{counting multiplicity}$$

The **defect of f with respect to Q by**

$$0 \leq \delta_f(Q) = \liminf_{r \to \infty} (1 - \frac{N_f(r,Q)}{(\deg Q)\, T_f(r)} \leq 1.$$

Question

1. What is the best upper bounded of

$$\sum_{i=1}^{q} \delta_f(Q_i) \leq ?$$

(This is a problem of the second main theorem)

2. If two nonconstant meromorphic functions which have the same inverse images of elements or sets. Then, will they be equal? (This is a problem of uniqueness)

Question 1: The problem of second main theorem

- **(I)** $n = 1$:

  - **For complex numbers:** The Classical Second Main Theorem of Nevanlinna: f is a meromorphic function, a_i are distinct points in $\mathbb{C}$. Then

$$\sum_{i=1}^{q}(\delta_f(a_i) + \theta_f(a_i)) \leq 2,$$

where

$$\theta_f(a) = \liminf_{r\to\infty} \frac{N_f(r,a) - \bar{N}_f(r,a)}{T_f(r)}.$$

- **For small functions:** f is a meromorphic function, α is called *small function with f* iff $T_\alpha(r) = o(T_f(r))$

- Steinmetz and Osgood proved: f is a meromorphic function, $\alpha_i, i = 1, ..., q$ are small functions with f, then

$$\sum_{i=1}^{q} \delta_f(\alpha_i) \leq 2.$$

- Yamanoi (Acta Math 2004): f is a meromorphic function, $\alpha_i, i = 1, ..., q$ are small functions with f, then

$$\sum_{i=1}^{q} (\delta_f(\alpha_i) + \theta_f(\alpha_i)) \leq 2.$$

- **(II)** $n \geq 2$:
  - **Cartan, Mathematica, 1933:** $f : \mathbb{C} \to \mathbb{P}^n$ is a linearly nondegenerate, $H_1, \ldots, H_q$ are hyperplanes such that any $n + 1$ of them is linearly independence. Then

$$\sum_{i=1}^{q} \delta_f(H_i) \leq n + 1.$$

- **Shiffman, Indiana U. Math. J., 1979:** $D_1, \ldots, D_q$ are hypersurfaces in general position with $\mathbb{P}^n$, $f : \mathbb{C}^m \to \mathbb{P}^n$ is a of finite order and $\mathrm{Im} f \not\subset D_i$ for any i. Then

$$\sum_{i=1}^{q} \delta_f(D_i) \leq 2n.$$

Definition

X is a n-dimensional projective variety. $D_1, \ldots, D_q$ of X is said to be **in general position** if

- the codimension of $\cap_{j=1}^{q} D_j$ in X is q when $q \leq n$;
- for any $\{i_0, \ldots, i_n\} \subset \{1, \ldots, q\}$, $\cap_{j=0}^{n} D_{i_j} = \emptyset$ when $q \geq n+1$.

- **Eremenko and Sodin, St. Petersburg Math. J., 1992:**

  $f : \mathbb{C} \to \mathbb{P}^n$ and $\mathrm{Im} f \not\subset D_i$. Then

$$\sum_{i=1}^{q} \delta_f(D_i) \leq 2n.$$

- **Ru, American J. Math., 2004:** $f : \mathbb{C} \to \mathbb{P}^n$ is a algebraically nondegenerate. Then

$$\sum_{i=1}^{q} \delta_f(D_i) \leq n+1.$$

  Ru's results need condition f is alg. nondegenerate and it is not clear if these results can implies Eremenko-Sodin's results which only need nondegenerate in the hypersufaces.

Conjecture (Griffiths' conjecture)

Assume $D_1, \ldots, D_q$ are hypersurfaces of degree d, in general position with $\mathbb{P}^n$ and $f : \mathbb{C} \to \mathbb{P}^n$ is a algebraically nondegenerate holomorphic map. Then

$$\sum_{i=1}^{q} \delta_f(D_i) \leq (n+1)/d.$$

(Infact Griffiths asked if the above inequality holds for any map which is algebraically nondegenerate in hypersuface of degree d. There is a conterexample in the case of $\mathbb{P}^2$, D_1, D_2, D_3 are the conics, normal crossing, then there exits a map f which is not degenerate of degree 2, but degenerate of degree 3 and $\sum_{i=1}^{3} \delta_f(D_i) \leq (n+1)/d$ $(= 7/2)$ say that it is not true).

Non-Archimedian case

- **K**: an algebraically closed field of arbitrary characteristic, complete with respect to a non-Archimedean absolute value $|\ |$.
- Let h is an entire function on **K**, then for each real number $r \geq 0$,

$$h(z) = \sum_{j=0}^{\infty} a_m z^m,$$

we define

$$\begin{aligned} |h|_r : &= \sup_j |a_j| r^j = \sup\{|h(z)| : z \in \mathbf{K}\ \mathit{with}\ |z| \leq r\} \\ &= \sup\{|h(z)| : z \in \mathbf{K}\ \mathit{with}\ |z| = r\}. \end{aligned}$$

- Let $f : \mathbf{K} \rightarrow \mathbb{P}^n(\mathbf{K})$ be a non-constant analytic curve in projective space with $\tilde{f} = (f_0, ..., f_n)$ be a reduced representative.

$$||f||_r := \max\{|f_0|_r, ..., |f_n|_r\},$$

$$T_f(r) := \log ||f||_r,$$

$$m_f(r, Q) := \log^+ \frac{||f||_r^d}{|Q \circ f|_r}.$$

$n = 1$ f is a Non-Archimedian meromorphic function, and

$a_i \in \mathbf{K}, i = 1, ..., q$. Then

$$(q-1)T_f(r) \leq \sum_{i=1}^{q} N_f(r, a_i) + O(1), \qquad (*)$$

$$(q-2)T_f(r) \leq \sum_{i=1}^{q} \bar{N}_f(r, a_i) - \log r + O(1), \qquad (*)$$

and hence

$$\sum_{i=1}^{q} \delta_f(a_i) \leq 1,$$

Question: What is the best bound in (*) when we consider a_i to be small functions?

Our first result is as follows.

Theorem

Let f be a nonconstant meromorphic function on $\mathbf{K}$. Let $a_1, \ldots, a_q$ $(q \geq 5)$ be q distinct small functions with respect to f. We have

$$\frac{2q}{5} T(r, f) \leq \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f - a_i}) + S(r, f).$$

Denote

$$\overline{E}(a,k,f)=\{z\in \mathbf{K}:\quad f(z)-a=0 \text{ with multiplicities at most } k.\}$$

A zero of $f-\infty$ means a pole of f.

Denote $\overline{E}(a,\infty,f)=\overline{E}(a,f)$.

We say that f and g *share a function a ignoring multiplicities* if

$\overline{E}(a,f)=\overline{E}(a,g)$.

Theorem

Let f and g be two nonconstant meromorphic functions on $\mathbf{K}$. *Let $a_1, \ldots, a_q \quad (q \geq 5)$ be q distinct small functions with respect to f and g. Let $k_1, \ldots, k_q$ be q positive integers or $+\infty$ with*

$$\sum_{j=1}^{q} \frac{1}{k_j + 1} < \frac{2q(q-4)}{5(q+4)}.$$

If

$$\overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \quad (j = 1, \ldots, q),$$

then $f \equiv g$.

In the case $k_1 = \cdots = k_q = k$, we can get the result with slightly smaller multiples as follows.

Theorem

Let f and g be two nonconstant meromorphic functions on $\mathbf{K}$. Let $a_1, \ldots, a_q \quad (q \geq 5)$ be q distinct small functions with respect to f and g. Let k be a positive integer or $+\infty$ with $k > \frac{3(q+4)}{2(q-4)}$. If

$$\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g) \quad (j = 1, \ldots, q),$$

then $f \equiv g$.

Corollary

Let f and g be two nonconstant meromorphic functions on $\mathbf{K}$. Let $a_1, \ldots, a_5$ be 5 distinct small functions with respect to f and g. If f and g share a_j ignoring multiplicities $(j = 1, \ldots, 5,)$ then $f \equiv g$.

Sketch of proof We first consider the following lemma.

Lemma

Let f be a nonconstant meromorphic function on $\mathbf{K}$. Let $a_1, \ldots, a_5$ be distinct small functions with respect to f. We have

$$2T(r,f) \leq \sum_{i=1}^{5} \overline{N}(r, \frac{1}{f-a_i}) + S(r,f).$$

- By the Lemma, for every subset $\{i_1, \ldots, i_5\}$ of $\{1, \ldots, q\}$ such that $1 \leq i_1 < \cdots < i_5 \leq q$, we have

$$2T(r,f) \leq \sum_{s=1}^{5} \overline{N}\big(r, \frac{1}{f - a_{i_s}}\big) + S(r,f). \qquad (1)$$

- The number of such inequalities is C_q^5. Taking summing up over all subsets $\{i_1, \ldots, i_5\}$ of $\{1, \ldots, q\}$:

$$2\mathrm{C}_q^5 T(r,f) \leq \sum_{\substack{\{i_1,\ldots,i_5\}\subset\{1,\ldots,q\} \\ 1\leq i_1<\cdots<i_5\leq q}} \Big(\overline{N}\big(r, \frac{1}{f-a_{i_1}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_2}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_3}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_4}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_5}}\big)\Big) + S(r,f)$$

for each index i_k, the number of terms $\overline{N}\big(r, \frac{1}{f-a_{i_k}}\big)$ is C_{q-1}^4.

Hence,

$$2\mathrm{C}_q^5 T(r,f) \leq \mathrm{C}_{q-1}^4 \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) + S(r,f).$$

It follows that

$$\frac{2q}{5} T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) + S(r,f).$$

- **Khoai and Tu, Internat. J. Math, 1995:** $H_1, ..., H_q$ are hyperplanes in general position, $f : \mathbf{K} \to \mathbb{P}^n(\mathbf{K})$ is non linear degenerate. Then

$$\sum_{i=1}^{q} \delta_f(G_i) \leq n+1.$$

- **Ru, Proc. A.M.S., 2001:** $G_1, \ldots, G_q$ are hypersurfaces of $\mathbb{P}^n(\mathbf{K})$ in general position, $f : \mathbf{K} \to \mathbb{P}^n(\mathbf{K})$ and $\mathrm{Im} f \not\subset G_i$ for any i. Then

$$\sum_{i=1}^{q} \delta_f(G_i) \leq n.$$

Theorem

X be a projective variety of dimension $n \geq 1$. Let $G_1, .., G_q$ be hypersurfaces in general position with X. Let $f : \mathbf{K} \to X$ be a non-constant analytic map and $Imf \not\subset G_i$ for any i. Then

$$\sum_{i=1}^{q} \delta_f(G_i) \leq n.$$

Corollary

With the assumptions and notation as in the theorem, there is at most n hypersurfaces G_j such that $\delta_f(G_j) > 0$.

Corollary

Let X be a projective variety of dimension n and $G_1, \cdots, G_q$ are hypersurfaces in general position. If $q \geq n+1$ then $f : \mathbf{K} \longrightarrow X \setminus \{\cup_{i=1}^{q} G_i\}$ is constant.

The Complex case

Definition

A manifold X over algebraically closed, and complete field F is said to be **hyperbolic** (in the sense of Brody) if every analytic map from F to X is constant.

Conjecture (Kobayashi - Zaidenberg Conjecture)

In the complex field, the complements of "generic" hypersurfaces in $\mathbb{P}^n$ with degree at least $2n+1$ are hyperbolic.

Previous Work

- **Green, Proc. A. M. S., 1977:**

  $\mathbb{P}^n(\mathbb{C}) \setminus \{ 2n+1$ hyperplanes in general position$\}$ is $\mathbb{C}$-hyperbolic.

- **Eremenko-Sodin, St. Petersburg M. J., 1992; and Ru, J. Reine Angew. Math., 1993]:** $\mathbb{P}^n(\mathbb{C}) \setminus \{ 2n+1$ hypersurfaces in general position$\}$ is $\mathbb{C}-$hyperbolic.

- **Dethloff, Schmacher, and Wong, Amer. J. Math., 1995; Duke Math. J.,1995:** For C_i is generic curves, $\mathbb{P}^2(\mathbb{C}) \setminus \cup_{i=1}^4 C_i$ is $\mathbb{C}-$hyperbolic and $\mathbb{P}^2(\mathbb{C}) \setminus \cup_{i=1}^3 C_i$ is $\mathbb{C}-$hyperbolic if $\deg C_i \geq 2$ for $i = 1, 2, 3$

Non-Archimedian cases

- **Ru, Proc. A.M.S., 2001:**

  $\mathbb{P}^n(\mathbf{K}) \setminus \{\, n+1$ hypersurfaces in general position$\}$ is $\mathbf{K}$-hyperbolic.

- **Proc. A. M. S. 135 (2007):**

  $X \setminus \{\, \dim X + 1$ hypersurfaces in general position$\}$ is $\mathbf{K}$-hyperbolic.

Question

Let $D_1,\ldots,D_q$, $q \le n$, be q distinct generic hypersurfaces in $\mathbb{P}^n(\mathbf{K})$. If $\sum_{i=1}^q \deg D_i \ge 2n$, then $\mathbb{P}^n \setminus \cup_{i=1}^q D_i$ is $\mathbf{K}$-hyperbolic.

Theorem (Wang, Wong and A., J. Number Theory 128 (2008))

Let X be an n-dimensional nonsingular projective variety. Let $D_i = \{P_i = 0\}$, $1 \leq i \leq q$, be hypersufaces in general position. Let $f : \mathbf{K} \longrightarrow X \setminus \cup_{i=1}^{q} D_i$. Then

$$codim(Im)f \geq \min\{n+1, q\} - 1.$$

In particularly, if $q \geq 2$ then f is algebraically degenerate, and if $q \geq n+1$ then $X \setminus \cup_{i=1}^{q} D_i$ is $\mathbf{K}$-hyperbolic.

The following example shows that the theorem is sharp.

Example

Let $X = \mathbb{P}^n$ and $q \leq n$ and $D_1 = \{X_{n-q+1} = 0\},\ldots,D_q = \{X_n = 0\}$. Let $f_0,\ldots,f_{n-q}$ be algebraically independent K-analytic functions. Then $f = (f_0, f_1, \ldots, f_{n-q}, 1, \ldots, 1) : \mathbf{K} \longrightarrow \mathbb{P}^n \setminus \cup_{i=1}^{q} D_i$, and $\text{codim}(\text{Im})f = q - 1$.

Lemma

Let C be a irreducible projective curve. Then $C \setminus \{two\ distinct\ points\}$ is $\mathbf{K}$-hyperbolic.

Results in projective spaces

Definition

Nonsingular hypersurfaces $D_1, ..., D_n$ in $\mathbb{P}^n(\mathbf{K})$ **intersect transversally** if for every point $x \in \cap_{i=1}^n D_i$, $\cap_{i=1}^n \Theta_{D_i,x} = \{x\}$, where $\Theta_{D_i,x}$ is the tangent space to D_i at x.

Theorem

Let $D_1, \ldots, D_n$ be nonsingular hypersurfaces in $\mathbb{P}^n(\mathbf{K})$ intersecting transversally. Then $\mathbb{P}^n \setminus \cup_{i=1}^n D_i$ is $\mathbf{K}$-hyperbolic if $\deg D_i \geq 2$ for each $1 \leq i \leq n$.

The assumption on the degree of the hypersurfaces is sharp.

Example

$D_1 = \{X_0 = 0\}$, and $D_i = \{X_0^2 + a_{i1}X_1^2 + \cdots + a_{in}X_n^2 = 0\}$ with $a_{i1} + \cdots + a_{in} = 0$ for $2 \leq i \leq n$ such that every $n-1$ by $n-1$ submatrix of the matrix $(a_{ij})_{i,j}$, $2 \leq i \leq n, 1 \leq j \leq n$, has rank $n-1$. Then these hypersurfaces intersect transversally. Clearly, the analytic map $f(z) = (1, z, z, \ldots, z)$ does not intersect any of the hypersurfaces D_i, $1 \leq i \leq n$.

The particular case when $n = 2$

Definition

Let D be a curve of degree $d \geq 3$ in $\mathbb{P}^2$. A nonsingular point x of D is said to be a **maximal inflexion point** if there exits a line intersecting D at x with multiplicity d.

Remark

The curve $X^d - YZ^{d-1} = 0$ has a maximal inflexion point $P = (0, 0, 1)$ if $d \geq 3$. Every smooth cubic has 9 maximal inflexion points counting multiplicities. Since a maximal inflexion point is an inflexion point, the coefficients of the defining equation of the curve need to satisfy an algebraic equation (i.e. its Hessian form cf. [**?**]). Therefore, it is not difficult to see that a generic curve of degree $d \geq 4$ has no maximal inflexion points.

Theorem

Let D_1 *and* D_2 *be nonsingular projective curves in* $\mathbb{P}^2$. *Assume that* D_1 *and* D_2 *intersect transversally and* $\deg D_1 \leq \deg D_2$. *Then* $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ *is* **K**-*hyperbolic if and only if either* $\deg D_1, \deg D_2 \geq 2$ *or* $\deg D_1 = 1$, $\deg D_2 \geq 3$ *and* D_1 *does not intersect* D_2 *at any maximal inflexion point.*

To prove the theorem, we first study some cases that

$\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ fails to be **K**-hyperbolic.

Lemma

*$\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not **K**-hyperbolic if*

(i) $\deg D_1 = 1$ *and* $\deg D_2 \leq 2$*;*

(ii) $\deg D_1 = \deg D_2 = 2$ *and* D_1 *and* D_2 *intersect tangentially;*

(ii) $\deg D_1 = 1$, $\deg D_1 \geq 3$ *and* D_1 *does intersect* D_2 *at a maximal inflexion point of* D_2*.*

The non-archimedean analogue of the Kobayashi-Zaidenberg conjecture for the case of $\mathbb{P}^2$ omitting two generic curves follows directly.

Corollary

Let D_1 and D_2 be distinct generic curves in $\mathbb{P}^2$. If $\deg D_1 + \deg D_2 \geq 4$ then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is $\mathbf{K}$-hyperbolic.